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MEASUREMENT OF RAPIDLY VARYING GAS TEMPERATURES IN AN UNSTEADY FLOW

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Abstract—For a thin thermometer having no radial temperature gradient but showing a certain thermal inertia the displacement of mean temperature related to the mean temperature of gas is determined together with the temperature record fundamental and higher harmonics in the case when both the measured temperature and the heat-transfer coefficient are varying periodically at the same frequency having different phase displacements. The effect of thermal conduction along the thermometer is taken into consideration.

NOMENCLATURE

- A, temperature variation amplitude;
- a, $\dfrac{1}{\tau_0} = \dfrac{2\alpha}{Rc\rho}$;
- c, specific heat of metal;
- k, thermal diffusivity, $= \dfrac{\lambda}{c \cdot \rho}$;
- l, length;
- n, $\dfrac{\omega}{a_0}$;
- R, radius of sensing element;
- t, temperature;
- x, distance along sensing element from the centre of span.

Greek symbols

- φ, phase angle;
- α, heat-transfer coefficient;
- λ, thermal conductivity;
- ρ, density of metal;
- τ, time;
- τ_0, time constant, $= \dfrac{Rc\rho}{2\alpha}$;
- ω, angular velocity.

1. INTRODUCTION

THE MEASUREMENT of rapidly varying gas temperatures by means of contact thermometers under the condition of simultaneous periodical variations of velocity of flow, density, viscosity and thermal conduction of gas, hence, under the conditions involving a variable heat-transfer coefficient has got a particular importance due to the very intensive actual research work on the transient processes of combustion and gas dynamics. In measurement practice, irrespective of typical errors encountered in the measurement of rapidly varying gas temperatures, some particular errors appear. The first attempt to determine those errors has been made by Gordov [1–6]. Considering the variation of temperature along the radius of a cylindrical sensing element and neglecting the thermal conduction along the element, Gordov has proved that in a medium in which both the temperature and heat transfer are harmonically variable, the measured mean temperature differs from the medium mean temperature, and the sensing element temperature pulsation curve consists of two component harmonics with different amplitudes and phase displacements, so that the curve does not represent the gas temperature pulsation. From Gordov's formula defining the magnitude of displacement of the temperature mean level one can conclude that this displacement maximum value as related to the temperature variation amplitude amounts to 0·25 relative pulsation of the heat-transfer coefficient. The maximum displacement occurs at the zero phase angle between the temperature pulsation and that of the heat-transfer coefficient; it disappears at the phase angle equal to $\pi/2$.

However, Kaganov and Rozenshtok [7] have proved that the results obtained by Gordov do not meet the boundary conditions due to errors in determining the integration constants. Applying the small parameter method to the problem of heat exchange with a variable heat-transfer coefficient, as discussed by Gordov, they have shown that in the case under consideration all harmonics at frequencies equal to the whole multiples of the fundamental one should appear in the temperature record, and that the maximum value of the mean temperature displacement should be equal to the gas temperature pulsation amplitude multiplied by a relative half amplitude of the heat-transfer coefficient pulsation. This value appears whenever the change of the heat-transfer coefficient has a certain phase lead in advance of the temperature change. Considering the numerous contradictions in the determined mean value and a very complex form of the results obtained up to now which neither permit the determination of higher harmonics nor fundamental component of the temperature fluctuation of the sensing element being of particular interest for the measurement of rapidly varying temperatures, the problem is to be revised on the base of some other assumptions.

Usually, a measurement of rapidly varying gas temperatures is performed by means of thin wire thermometers or thermocouples as represented in Fig. 1. The radial temperature gradient can be neglected in those thermometers due to the very short time of thermal diffusion in this direction $\tau = R^2/k$ (where k is the thermal diffusivity, R being the sensing element radius). However, the thermal conduction along the sensing element to the supporting electrodes cannot be neglected. Generally, such a sensing element must be very short because of tensile stress (proportional to the length–diameter ratio). So, the starting point to determine the errors in measuring varying temperatures at a variable heat-transfer coefficient should be an analysis of behaviour of a thin thermometer

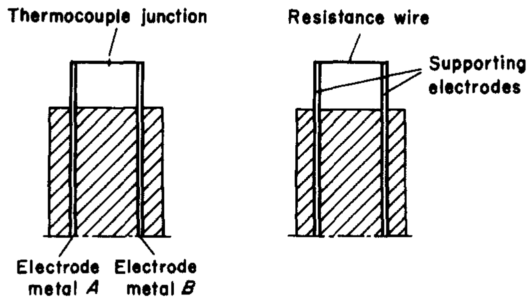

FIG. 1. Thermometers used for measurement of rapidly varying temperatures.

taking into consideration the effect of sensing element supports. This is the problem discussed in this paper.

2. DIFFERENTIAL EQUATION

The equation of temperature variations for a thin thermometer may be written as follows:

$$\frac{\partial t}{\partial \tau} = k\frac{\partial^2 t}{\partial x^2} + \frac{1}{\tau_0(\tau)}(t_g - t) \tag{1}$$

where

- t, temperature of the thermometer wire in the point defined by the coordinate x;
- τ, time;
- k, thermal diffusivity;
- t_g, gas temperature;
- $\tau_0(\tau)$, time constant defined by

$$\frac{1}{\tau_0(\tau)} = \frac{2\alpha(\tau)}{c\rho R};$$

where

- $\alpha(\tau)$, heat-transfer coefficient;
- c, specific heat of metal;
- R, radius of sensing element;
- ρ, density of metal.

A general solution of the equation (1) may be obtained with the method of separation of variables. In this way we obtain a complicated result suitable for numerical treatment only, even for the simple functions $1/\tau_0(\tau)$ and $t_g(\tau)$.

Considering the foregoing to obtain more general conclusions some separate simplified problems must be discussed, i.e. responses of a thermometer of infinite length followed by the estimation of the effect of the sensing element supports on the measurement results.

3. RESPONSE OF AN INFINITELY LONG THERMOMETER

If the effect of supports is neglected, the equation (1) can be written as follows:

$$\frac{\partial t}{\partial \tau} + a(\tau)(t - t_g) = 0 \tag{2}$$

where

$$a(\tau) = \frac{1}{\tau_0(\tau)}.$$

The solution is to be discussed at

$$\begin{aligned} a(\tau) &= a_m + \Delta a \sin \omega\tau \\ t_g(\tau) &= t_m + A \sin(\omega\tau + \varphi) \end{aligned} \tag{3}$$

for the initial condition $t(0) = t_0$.

In that case the general solution of the equation (2) is as follows:

$$t = e^{-F}\left[t_0 + \int_0^\tau a(\tau)\, t_g(\tau)\, e^F \,d\tau\right] \tag{4}$$

where

$$F(\tau) = \int_0^\tau a(\tau)\, d\tau.$$

Unfortunately, the integral written in the equation (4) cannot be expressed in a finite form for the variations of both the heat-transfer coefficient and the gas temperature defined in the relations (3). Attempts have been made to solve this integral in a form of a series by series expansion of the term e^F. However, at the very beginning of expansion, the first terms of the obtained solution have appeared as very complex ones, moreover, the series has been found not sufficiently convergent to make the solution method useful. Therefore, two other methods have been applied to obtain the approximate solutions: a method of selection of functions (3), to get the solution of the integral (4) as simple as possible, and the method of successive approximations based on the assumption that the heat-transfer coefficient variations are small.

3.1. *Method of selection of heat-transfer coefficient variations*

The solution of the integral written in the equation (4) appears relatively simple assuming that

$$a(\tau) = a_m + \frac{d\omega \sin \omega\tau}{1 - d\cos\omega\tau} \tag{5}$$

and that the rest of the conditions remain unchanged. That simplicity of the solution is due to the fact that in the right hand side of the expression (5) the numerator is a derivative of the denominator, so that the term e^F can be expressed simply as

$$e^{a_m\tau} \cdot \frac{1 - d\cos\omega\tau}{1 - d}.$$

Since the true variations of the heat-transfer coefficient involving the variations of a are never exactly harmonic, the foregoing assumption of non-exactly sinusoidal variations is fully justified. In order to prevent excessive departures from the sinusoidal variations of heat-transfer coefficient it is necessary to satisfy the condition of $d \ll 1$. On the other hand the heat-transfer coefficient cannot be negative, then $d\omega/a_m < 1$. Both conditions mentioned above represent a limitation of the applications of the method: viz. for a thermometer with a small thermal inertia ($\omega/a \approx 1$) the solution may be obtained merely for very small pulsations of a. On the contrary, for a thermometer with a significant thermal inertia the method in question enables us to analyse the effect of considerable variations of the heat-transfer coefficient on the thermometer response. For the accepted assumptions, after having neglected the expressions disappearing against time, and the terms of order d^2 the solution of the equation (2) takes the following form

$$t = t_m - \tfrac{1}{2}Adn\frac{n}{\sqrt{(1+n^2)}}\sin(\varphi - \arctan n) + A\frac{1}{\sqrt{(1+n^2)}}\sin(\omega\tau + \varphi - \arctan n) + Adn\frac{n\sqrt{(n^4 + \frac{5}{4}n^2 + \frac{1}{4})}}{(1+n^2)(1+4n^2)}\sin\left(2\omega\tau + \varphi + \arctan\frac{3n}{2n^2 - 1}\right). \quad (6)$$

In this solution n denotes the relation ω/a_m, i.e. a value equal to the product of frequency and the sensing time constant of the element for average conditions of heat transfer.

3.2. *Method of successive approximations*

In order to obtain the results valid within a wider measuring range, in particular applicable to small inertia thermometers, the equation (2) is solved additionally with the aid of the method of successive approximations.

Equation (2) can be written as follows:

$$\frac{\partial t}{\partial \tau} + a_m t = (a_m + \Delta a \sin\omega\tau)\, t_g - \Delta a\, t \sin\omega\tau. \quad (7)$$

When Δa is small, the solution of the equation (7) without the last term on the r.h.s. can be accepted as the first approximation. Let us denote this solution as t_1. The next successive approximation t_2 is obtained by forming a sum of t_1 and the solution of the following equation

$$\frac{\partial t_{02}}{\partial \tau} + a_m t_{02} = -\Delta a\, t_1 \sin\omega\tau. \quad (8)$$

Further approximations are obtained in an analogous way

$$\frac{\partial t_{03}}{\partial \tau} + a_m t_{03} = -\Delta a t_2 \sin\omega\tau$$
$$t_3 = t_1 + t_{03}, \text{ etc.} \quad (9)$$

The solution of equation (2) after the second approximation obtained with the aid of the method described above takes the following form:

$$t_2 = t_m - \tfrac{1}{2}A\frac{\Delta a}{a_m}\frac{n}{\sqrt{(1+n^2)}}\sin(\varphi - \arctan n) + A\frac{1}{\sqrt{(1+n^2)}}\sin(\omega\tau + \varphi - \arctan n) + A\frac{\Delta a}{a_m}\frac{n\sqrt{(n^4 + \frac{5}{4}n^2 + \frac{1}{4})}}{(1+n^2)(1+4n^2)}\sin\left(2\omega\tau + \varphi + \arctan\frac{3n}{2n^2 - 1}\right). \quad (10)$$

Then, the solution is identical with (6) provided the product dn is replaced by $\Delta a/a_m$. This fact, being a rather surprising case, reveals some interesting properties of the applied methods. The method of successive approximations is shown to be equivalent to the assumption of a certain deformation of the variable coefficient pulsations. If a slight deformation of that kind is taken as a criterion for the correct solution, one can conclude that the method of successive approximations, applied within a certain range, enables us to obtain accurate results even for considerable variations of the coefficient ($n \gg 1$ in the solution from 3.1). Within other ranges the admissible level of the variable coefficient changes is very low ($n < 1$ in the solution from 3.1).

Another view of the problem consists of taking a definite level of the changes in the variable coefficient in equation (2) as a correctness criterion, while the method of successive approximations is applied to solve the equation (2).

In this case one can state that within some ranges only very small deformations of the pulsations of the variable coefficient are admitted ($n \gg 1$), while within other ranges even large deformations ($n < 1$) do not affect the solution accuracy. Generally, the possibility of the appearance of such phenomena is not taken into account whenever the described methods are applied.

The solutions (6) and (10) determine the magnitude of the displacement of the mean level in relation to the gas mean temperature and they give two component harmonics of the sensing element temperature variations. When the method of successive approximations is applied, some higher harmonics may appear similarly as it has been shown in [7]. So, in the n approximation the component harmonics appear at frequency equal to the fundamental one multiplied by n. Since only two harmonics appear in the solution (6), this solution being valid for variations of $a(\tau)$, slightly deformed as compared with the sinusoidal ones, one may conclude that the first two harmonics should be much greater than the remaining ones, and that, qualitatively the results of Gordov, Kaganov, Rozenshtok and this work overlap in the case considered.

The essential feature of the solutions obtained consists in the property of linear dependence of all terms on the amplitude of the temperature pulsation.

The magnitude of displacement of the mean temperature level of the sensing element in relation to the mean temperature of the gas depends on both the phase displacement φ and the parameter n. For $\varphi = \arctan n$ no displacement appears, but for $\varphi = 0$ and $n \to \infty$ it proceeds to the maximum value $\frac{1}{2}A\,\Delta a/a_m$, i.e. to $\frac{1}{2}x$ product of the temperature variation amplitude and the heat-transfer coefficient relative pulsation. This result is compatible with the conclusions of the work by Kaganov and Rozenshtok. An interesting, simple, but in no case evident result is that the amplitude of the temperature record first harmonic does not depend upon the heat-transfer coefficient variations, being the same as for a thermometer put into a medium at constant heat-transfer coefficient equal to the variable coefficient mean value. As it results from the solutions for high values of n, the second harmonic amplitude amounts to $\frac{1}{4}x$ product of both the first harmonic and the amplitude of heat-transfer coefficient relative variations.

It is noteworthy that the phase displacements of both component harmonics are expressed very simply as a function of n depending additively on φ. It is worth noting that the solution for the second harmonic is probably more inaccurate than for the first because the influence of the distortion of the heat-transfer coefficient variations and second order elements may be stronger for this harmonic.

For a more detailed representation of obtained results, Figs. 2–5 show the mean level displacement, first and second harmonic amplitudes and the phase displacement plotted against n and φ. As a matter of particular interest one may point out that the relative maximum of the temperature record second

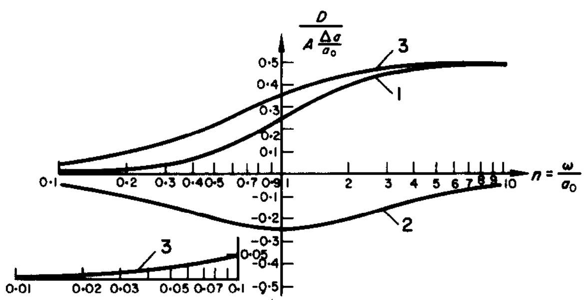

FIG. 2. Displacement of the thermometer temperature mean level related to the mean temperature of gas.

1. $\varphi = 0$
2. $\varphi = 90°$
3. Maximum displacement.

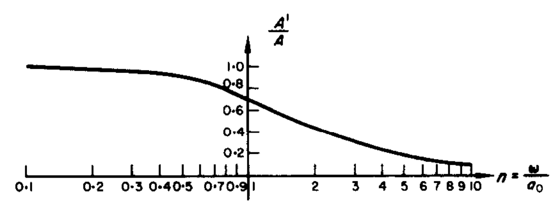

FIG. 3. Amplitude of the first harmonic of the thermometer temperature variations.

harmonic appears for small thermal inertia of the thermometer, i.e. for $n \approx 0{\cdot}7$. The maximum displacement of the mean level plotted against n is also noteworthy. When assessing

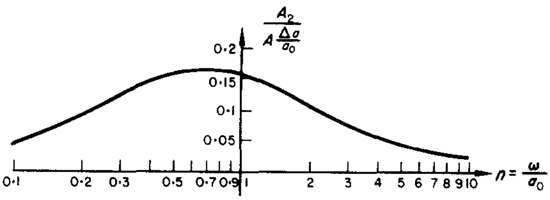

FIG. 4. Amplitude of the second harmonic of thermometer temperature variations.

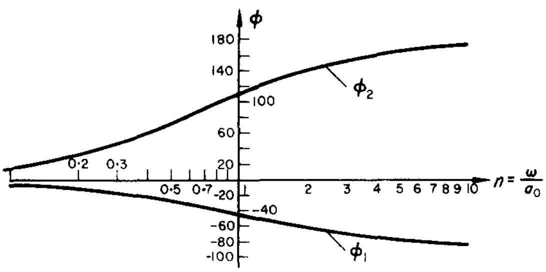

FIG. 5. Phase displacement of the first and second harmonics of the thermometer temperature variations for $\varphi = 0°$. To obtain phase displacements for other angles φ the curves must be shifted in parallel through the angle φ.

the obtained results generally, one must state that the heat-transfer coefficient variations involve a considerable distortion of the sensing element temperature variation in relation to the ambient temperature over a wide range of the conditions of measurement.

4. ESTIMATION OF THE EFFECT OF THE SENSING ELEMENT SUPPORTING ELECTRODES ON THE MEASUREMENT RESULTS

After a discussion of the physical side of the problem one may conclude that a variation of the heat-transfer coefficient should involve a change of the temperature distribution along the sensing element. These variations are characterized by the following characteristic time:

$$\tau_1 = \frac{l^2}{k} \tag{11}$$

where

l is the sensing element length,
k being the thermal diffusivity.

In connection with this time value one may distinguish between two limiting cases for which the estimation of the effect of the supports on the measurement results is relatively simple:

1. Whenever the characteristic time of thermal diffusivity along the span due to the thermal conduction is much shorter than the shortest lapse of time appearing in the heat-transfer coefficient pulsation, the variations are quasi-static and the temperature distribution corresponding to one value of the heat-transfer coefficient passes through several steady states until it corresponds to the heat-transfer coefficient's other value.
2. As the characteristic time is much greater than the longest lapse of time encountered in the observed variations of the heat-transfer coefficient, those variations cannot influence the temperature distribution, so that the effect of the supports may be determined similarly as for the case of a constant heat-transfer coefficient equal to its mean value within a defined period.

The solution of the equation (1) for the first case and for the boundary conditions given beneath:

$$\frac{\partial t_x}{\partial x} = 0 \qquad \text{for} \quad x = 0$$

and

$$t_x = t_p \qquad \text{for} \quad x = 1$$

will be as follows:

$$t_{x_1} = t_\infty(\tau)\left[1 + \frac{[t_p - t_\infty(\tau)]\cos\text{h}\, x\sqrt{\left(\frac{2\alpha(\tau)}{R\lambda}\right)}}{t_\infty(\tau)\cos\text{h}\, l\sqrt{\left(\frac{2\alpha(\tau)}{R\lambda}\right)}}\right] \tag{12}$$

where,

$t_\infty(\tau)$, temperature of a cylindrical sensing element of infinite length;
R, radius of sensing element;
l, sensing element half length;
t_p, temperature of supports;
$\alpha(\tau)$, heat-transfer coefficient;
λ, thermal conductivity.

Then, the maximum measuring error due to the effect of supports in accordance with the relation

$$\zeta = \frac{t_\infty - 1/l \int_0^l t(x)\,\mathrm{d}x}{t_\infty} \tag{13}$$

will be

$$\zeta_1(\tau) = \frac{t_p - t_\infty}{t_\infty} \cdot \frac{\tan\mathrm{h}\, l \sqrt{\left(\frac{2\alpha(\tau)}{R\lambda}\right)}}{l \sqrt{\left(\frac{2\alpha(\tau)}{R\lambda}\right)}} \tag{14}$$

hence, for often encountered conditions:

$$\zeta_1(\tau) = \frac{t_p - t_\infty}{t_\infty}\left[l \sqrt{\left(\frac{2\alpha(\tau)}{R\lambda}\right)}\right]^{-1}. \tag{15}$$

The solution of the equation (1) for the second case and for the boundary conditions:

$$\frac{\partial t_x}{\partial x} = 0 \quad \text{for} \quad x = 0$$

and

$$t_x = 0 \quad \text{for} \quad x = 1$$

for the ambient temperature harmonic variations expressed as $t_g = A\,\mathrm{e}^{i\omega t}$ will be:

$$t_{x_2} = t_\infty\left[1 - \frac{\cos\mathrm{h}\, x \sqrt{\left(\frac{2\alpha_m}{R\lambda}(1 + i\,\omega\tau_{0m})\right)}}{\cos\mathrm{h}\, l \sqrt{\left(\frac{2\alpha_m}{R\lambda}(1 + i\,\omega\tau_{0m})\right)}}\right]. \tag{16}$$

The measuring error due to the effect of the supports defined in accordance with the relation (13) will be approximately:

$$\zeta_2 = \{l\,[\sqrt{(2\alpha_m/R\lambda)}]\,(1 + \omega^2\tau_{0m}^2)^{\frac{1}{4}}\}^{-1} \tag{17}$$

where:

τ_{0m} is the time constant of the sensing element for the mean value of the heat-transfer coefficient. In the foregoing consideration the errors due to a difference between the sensing element mean temperatures and those of the supports have been neglected. Those errors can be evaluated using the relation (15).

However, in many cases some difficulties arise because neither the condition (1) nor (2) can be met. In such cases the transition from one state of the sensing element temperatures to another would occur in a rather complex manner with the generation of a kind of thermal wave which would propagate along the length of the sensing element and would influence continuously the effect of the supports on the measurement result.

To determine this influence it is necessary to solve the equation (1) without simplifying it. However, as it has been mentioned, such a solution is very complicated Nevertheless, both the highest and lowest value of this error can be estimated. The probable error will be approximately equal to the arithmetic mean of those values.

Intuitively* it seems quite evident that the minimum effect of the supports would appear in the case in which the assumed heat-transfer coefficient would be constant and equal to the maximum value within the whole cycle, and, conversely, that the maximum effect would appear at the minimum value of the heat-transfer coefficient.

In accordance with the relation (17) one may write for this particular case the following

* Despite relatively considerable progress in the theory of parabolic differential inequalities [8, 9] a corresponding mathematical proof is still lacking.

expression:

$$\zeta_3 = \frac{1}{2}\,\frac{1}{l\sqrt{(2/R\lambda)}}\left[\frac{1}{(\sqrt{\alpha_{\max}})(1+\omega^2\tau_{0\max}^2)^{\frac{1}{2}}} + \frac{1}{(\sqrt{\alpha_{\min}})(1+\omega^2\tau_{0\min}^2)^{\frac{1}{2}}}\right] \quad (18)$$

where the subscripts min and max denote respectively the minimum and maximum values of the particular parameters.

5. CONCLUSIONS

1. The foregoing analysis proves that the mean temperature of the sensing element in the case of a variable heat-transfer coefficient is displaced in relation to the mean temperature of the medium. The highest value of this displacement equals one half the amplitude of the gas temperature variations multiplied by the amplitude of the relative pulsations of the heat-transfer coefficient. Although the maximum displacement appears for high values of the thermometer thermal inertia, this displacement is also substantial for thermometers showing small thermal inertia.
2. The variations of the gas temperatures as recorded by the thermometer are subject to great deformations. Whenever sinusoidal variations are measured, higher harmonics appear, among which the second harmonic is particularly important. The first harmonic amplitude does not depend on the heat-transfer coefficient variations, being the same as for a thermometer put into a medium having the constant heat-transfer coefficient equal to the variable coefficient mean value. For thermometers showing a significant thermal inertia the second harmonic amplitude amounts approximately to $\frac{1}{4}x$ product of both the first harmonic amplitude and the amplitude of the relative variations of the heat-transfer coefficient.
3. For the majority of cases the error due to the effect of the supports for a variable heat-transfer coefficient is identical with that appearing in a case of a constant heat-transfer coefficient equal to the mean value of the variable coefficient.

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Résumé—Pour un thermomètre fin n'ayant aucun gradient de température radial mais possédant une certain inertie thermique, le déplacement de la température moyenne en relation avec la température moyenne du gaz est déterminé en même temps que le fondamental et les harmoniques supérieurs de l'enregistrement de la température, dans le cas où la température mesurée et le coefficient de transport de chaleur varient tous les deux périodiquement à la même fréquence et avec différents déplacements de phase. L'effet de la conduction thermique le long du thermomètre est pris en considération.

Zusammenfassung—Für ein dünnes Thermometer ohne radialen Temperaturgradienten aber mit einer gewissen thermischen Trägheit wird die Abweichung der mittleren Temperatur in Abhängigkeit von der mittleren Gastemperatur bestimmt zusammen mit den fundamentalen und höheren harmonischen Temperaturschwankungen für den Fall, dass sich sowohl die gemessene Temperatur als auch der Wärmeübergangskoeffizient bei gleicher Frequenz und verschiedener Phasenabweichung periodisch ändert. Der Einfluss der Wärmeleitung im Thermometer wird berücksichtigt.

Аннотация—Для тонкого термометра, не имеющего радиального температурного градиента, но имеющего определенную тепловую инерцию, смещение средней температуры, отнесенной к средней температуре газа определяется вместе с показаниями термометра и высшими гармониками в случае, когда обе измеренные температуры и коэффициент теплообмена изменяются периодически с одинаковой частотой, но с различными сдвигами фаз. Также рассмотрен эффект теплопроводности вдоль термометра.